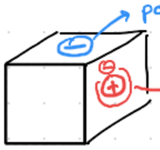
5) Frequency dispersion

1. Maxwell’s equations: from vaccuum to materials



possible external charges density ρ_{ext}

atoms = linked charges density ρ_ℓ

piece of material

1) Maxwell's equations in vaccuum

$$(i)\ \vec{\nabla}\cdot(\varepsilon_0\vec{\mathcal{E}}) = \rho_{ext} + \rho_\ell \quad (ii)\ \vec{\nabla}\cdot\vec{\mathcal{B}} = 0 \quad (iii)\ \vec{\nabla}\times\vec{\mathcal{E}} = -\frac{\partial\vec{\mathcal{B}}}{\partial t} \quad (iv)\ \vec{\nabla}\times\frac{\vec{\mathcal{B}}}{\mu_0} = \vec{j}_\ell + \vec{j}_{ext} + \varepsilon_0\frac{\partial\vec{\mathcal{E}}}{\partial t}$$

2) Average over a volume $\delta V(\vec{r})$ such that

$$a^3 \ll \delta V(\vec{r}) \ll \lambda^3$$

a: distance between atoms; λ: characteristic length of field variations

$$\vec{\mathcal{E}}, \vec{\mathcal{B}} \rightarrow \langle\vec{\mathcal{E}}\rangle, \langle\vec{\mathcal{B}}\rangle \qquad \rho_{ext}, \rho_\ell, \vec{j}_{ext}, \vec{j}_\ell \rightarrow \langle\rho_{ext}\rangle, \langle\rho_\ell\rangle, \langle\vec{j}_{ext}\rangle, \langle\vec{j}_\ell\rangle$$

$\langle\vec{\mathcal{E}}\rangle = 0$

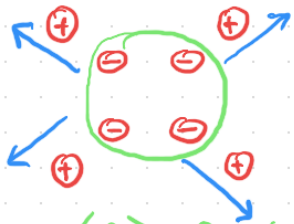
$\langle\rho_\ell\rangle = 0$

electrically neutral

$\langle\vec{\mathcal{E}}\rangle = E_0\vec{z}$ uniform

$\langle\rho_\ell\rangle = 0$

$\langle\vec{\mathcal{E}}\rangle$ non uniform



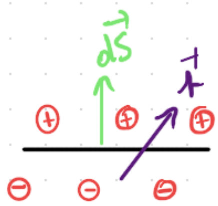
$\langle\rho_\ell\rangle = \rho_{ind} \neq 0$

induced charge density

$\Rightarrow$ we'll drop the $\langle\ \rangle$ for short notations.

3) Dipolar moment per unit volume : $\vec{P} = N\, q\, \vec{d}$

N: # atoms per unit volume; q: charge of nucleus; $\vec{d}$: average charge separation : from $\ominus$ to $\oplus$



charges through S = $\vec{P}\cdot\vec{dS}$

For a closed volume $Q_{ind} = \int_V \rho_{ind}\, dV = -\oint \vec{P}\cdot\vec{dS} = -\int \vec{\nabla}\cdot\vec{P}\, dV \Rightarrow \boxed{\rho_{ind} = -\vec{\nabla}\cdot\vec{P}}$

polarization $\vec{P}$

4) Conservation of induced charge $\Rightarrow \dfrac{\partial \rho_{ind}}{\partial t} + \vec{\nabla}\cdot\vec{J}_{ind} = 0 \Rightarrow \vec{\nabla}\cdot\left(\vec{J}_{ind} - \dfrac{\partial \vec{P}}{\partial t}\right) = 0$

$$\Rightarrow \exists\ \vec{m}\ /\ \boxed{\vec{J}_{ind} - \frac{\partial \vec{P}}{\partial t} = \vec{\nabla}\times\vec{m}}$$ magnetization $\vec{m}$

$\Rightarrow$ (i) $\vec{\nabla}\cdot(\varepsilon_0\vec{E} + \vec{P}) = \rho_{ext}$

(ii) $\vec{\nabla}\cdot\vec{B} = 0$

(iii) $\vec{\nabla}\times\vec{E} = -\partial_t\vec{B}$

(iv) $\vec{\nabla}\times\underbrace{\left(\frac{\vec{B}}{\mu_0} - \vec{m}\right)}_{\text{new } \vec{H}} = \vec{J}_{ext} + \partial_t\underbrace{(\varepsilon_0\vec{E} + \vec{P})}_{\text{new } \vec{D}}$

$\Rightarrow$ same as in vacuum

but: $\begin{cases} \vec{D} = \varepsilon_0\vec{E} + \vec{P} \\ \vec{B} = \mu_0\vec{H} + \vec{m} \end{cases}$

$(\rho_{tot}, \vec{J}_{tot}) \rightarrow (\rho_{ext}, \vec{J}_{ext})$

mesoscopic average

[[bonus: meaning of $\vec{m}$

$\vec{H}_{ext} \Rightarrow \vec{J}_{ind} \Rightarrow \vec{m}\neq 0$ inside the magnetic object $\Rightarrow \vec{B} = \mu_0\vec{H}_{ext} + \vec{m}$

At low frequencies, we can have $\left|\dfrac{\partial \vec{P}}{\partial t}\right| \ll \vec{J}_{ind}$ and then $\vec{J}_{ind} = \vec{\nabla}\times\vec{m}$

$$\boxed{\vec{M} = \frac{1}{2}\int_V (\vec{r}\times\vec{J}_{ind})\,dV = \int \vec{m}\,dV}$$

magnetic moment of object — magnetization

Proof: $\vec{M} = \dfrac{1}{2}\displaystyle\int_V \vec{r}\times(\vec{\nabla}\times\vec{m})\,dV = \dfrac{1}{2}\underbrace{\int_S \vec{r}\times(d\vec{S}\times\vec{m})}_{=0 \text{ outside on } S} - \dfrac{1}{2}\int((\vec{m}\times\vec{\nabla})\times\vec{r})\,dV$

but $(\vec{m}\times\vec{\nabla})\times\vec{r} = -\vec{m}\underbrace{(\vec{\nabla}\cdot\vec{r})}_{3} + \vec{m} = -2\vec{m}$ [QED]

When $\dfrac{\partial \vec{P}}{\partial t}$ is not negligible, the magnetic susceptibility model loses its meaning.

$\Rightarrow \mu$ makes no sense at high f $\Rightarrow$ better set $\mu = \mu_0$]]

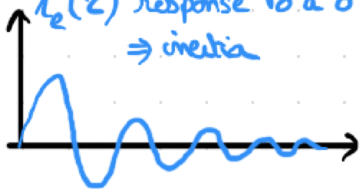
2. Permittivity in local dispersive media

$$\vec{\mathcal{D}}(t) = \varepsilon_0 \vec{\mathcal{E}}(t) + \vec{p}(t) \qquad \vec{\mathcal{B}}(t) = \mu_0 (\vec{\mathcal{H}} + \vec{m})$$

* non dispersive linear model : $\begin{cases} \vec{p}(t) = \varepsilon_0 \chi_e \vec{\mathcal{E}}(t) \\ \vec{m}(t) = \chi_m \vec{\mathcal{H}}(t) \end{cases}$ electric & magnetic susceptibilities

$$\Rightarrow \vec{\mathcal{D}} = \varepsilon_0 \underbrace{(1+\chi_e)}_{\varepsilon_r} \vec{\mathcal{E}} \qquad \vec{\mathcal{B}} = \mu_0 \underbrace{(1+\chi_m)}_{\mu_r} \vec{\mathcal{H}}$$

$\tilde{\chi}_e(\tau)$ response to a δ $\Rightarrow$ inertia



* Dispersive : $p(t) = \int_{0 \,\leftarrow\, \text{causal}}^{+\infty} \tilde{\chi}_e(\tau) \vec{\mathcal{E}}(t-\tau)\, d\tau$

$$\Rightarrow P(\omega) = \chi(\omega) E(\omega) \quad \text{with} \quad \chi(\omega) = \int_0^{+\infty} \tilde{\chi}_e(\tau) e^{-j\omega\tau} d\tau$$

$$\Rightarrow \varepsilon(\omega) = 1 + \chi_e(\omega)$$

Since $\tilde{\chi}_e(\tau)$ must be real : $\chi_e(-\omega) = \chi_e^*(\omega) \Rightarrow \boxed{\varepsilon(-\omega) = \varepsilon^*(\omega)}$

crossing condition

3. Lorentz model

$q = -e$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\Rightarrow m\ddot{\vec{r}} = -e(\vec{E} + \dot{\vec{r}} \times \vec{B}) - m\omega_0^2 \vec{r} - m\gamma\dot{\vec{r}}$$

$$\frac{\text{elec force}}{\text{mag force}} \simeq \frac{|E|}{\mu_0 |H v|} \simeq \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{\mu_0 v} = \frac{c_0}{v} \gg 1 \quad \text{(non relativistic speeds)}$$

$e^{j\omega t}$ time convention:

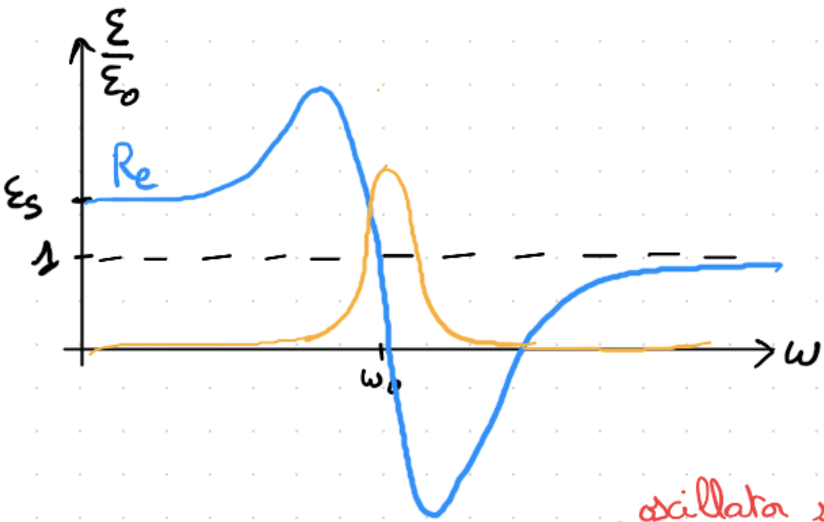
$$-m\omega^2 \vec{r} = -m\omega_0^2 \vec{r} - m\gamma j\omega \vec{r} - e\vec{E}$$

$$\Rightarrow \vec{r} = \frac{-e\vec{E}}{m(\omega_0^2 - \omega^2 + j\omega\gamma)} \quad \Rightarrow \vec{P} = Nq\vec{r} = \frac{Ne^2}{m(\omega_0^2 - \omega^2 + j\omega\gamma)} \vec{E}$$

Noting $\omega_p^2 = \dfrac{Ne^2}{m\varepsilon_0}$ we get $\boxed{\chi_e = \dfrac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\gamma}}$ Lorentzian response

$$\boxed{\varepsilon_r(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\gamma}}$$

Lorentz permittivity model



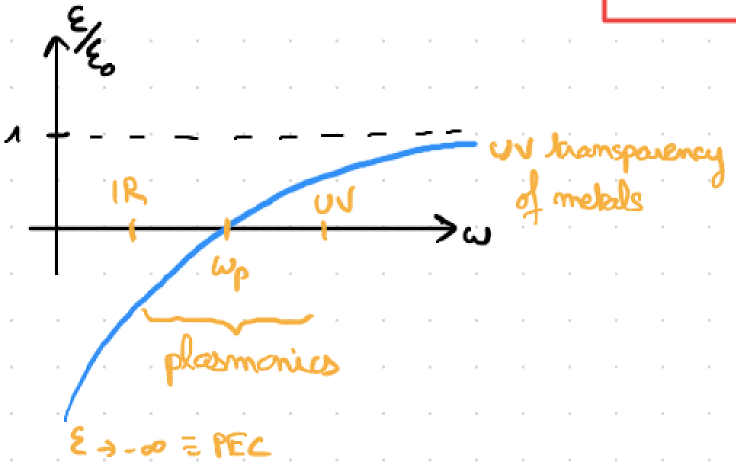
oscillator strength

$$\boxed{\varepsilon = \varepsilon_0 + \frac{Ne^2}{m} \sum_i \frac{f_i}{\omega_{0_i}^2 - \omega^2 + j\omega\gamma_i}}$$

generalized Lorentz oscillator model

4. Drude model

For free electrons (metals), $\omega_0 = 0$, $\boxed{\varepsilon = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 - j\omega\gamma}\right)}$ DRUDE model



The plot is made for $\gamma = 0$.

If $\omega = \omega_p$, $\varepsilon = 0$, $\lambda \to +\infty$

5. Debye model

For metals at low frequency : $\omega \ll \gamma$, $\omega_0 = 0$, and we get

$$\boxed{\varepsilon = \varepsilon_0 + \frac{Ne^2}{j\omega\gamma m} = \varepsilon_0 - j\frac{\sigma}{\omega} \qquad \sigma = \frac{Ne^2}{m\gamma} \text{ conductivity}}$$

DEBYE model

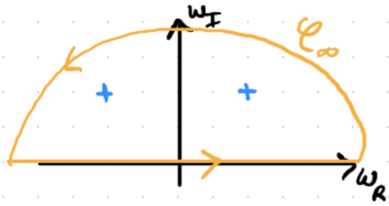
$\to$ the $\vec{J} = \sigma \vec{E}$ model for conductors

6. Causal bounded responses

Let's look at what impulse response the Lorentz model corresponds to:

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\gamma\omega} \Rightarrow \chi(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \chi(\omega)\, e^{j\omega\tau}\, d\omega$$

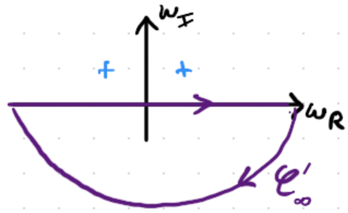
$\hookrightarrow$ 2 poles $\omega_{1,2} = \frac{j\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$



When $\tau > 0$, we calculate the integral using Cauchy residue theorem on $\mathcal{C}$

$$\oint_{\mathcal{C}} \chi(\omega)\, e^{j\omega\tau}\, d\omega = 2\pi j \sum \text{Res}\ \omega_{1,2} = \int_{-\infty}^{+\infty} \chi(\omega)\, e^{j\omega\tau}\, d\omega + \int_{\mathcal{C}_\infty} \underbrace{\chi(\omega)}_{\to 0}\, e^{j\omega_R\tau}\, \underbrace{e^{-\omega_I\tau}}_{\to 0}$$

$$\Rightarrow \chi(\tau) = \omega_p^2\, e^{-\gamma\frac{\tau}{2}}\, \frac{\sin\left(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\, t\right)}{\sqrt{\omega_0^2 - \gamma^2/4}}$$

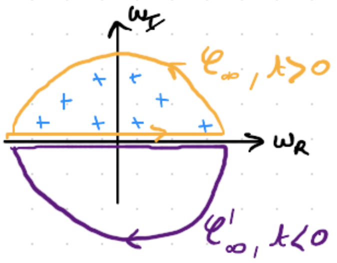


When $\tau < 0$, we do the same on $\mathcal{C}'$

$\Rightarrow e^{-\omega_I\tau} \to 0$ on $\mathcal{C}'_\infty$ but no poles $\Rightarrow \chi(\tau) = 0$

$$\Rightarrow \boxed{\chi(\tau) = \theta(\tau)\, \omega_p^2\, e^{-\gamma\frac{\tau}{2}}\, \frac{\sin\left(\sqrt{\omega_0^2 - \gamma^2/4}\, t\right)}{\sqrt{\omega_0^2 - \gamma^2/4}}}$$

Generalization



assuming $|\chi(\omega)|$ bounded

$$\oint \chi(\omega)\, e^{j\omega t}\, d\omega = 2\pi j \sum (\text{Res})$$

$e^{j\omega_R t}\, e^{-\omega_I t}$

- for $t>0$, goes to 0 when $\omega_I \to +\infty$
- for $t<0$, goes to 0 when $\omega_I \to -\infty$

poles on upper half complex plane only $\Rightarrow$ system is causal

$\exists$ pole on lower half $\mathbb{C}$ plane $\Rightarrow$ non causal system

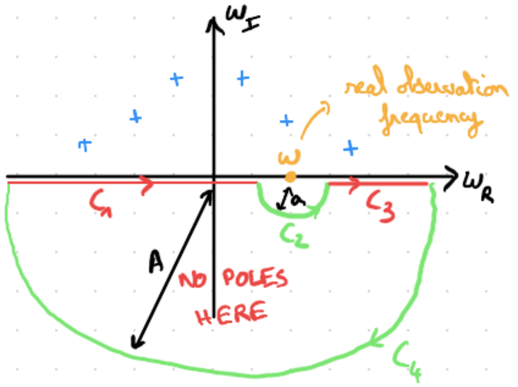
$\Rightarrow$ For any system to be causal, the poles of the (bounded) transfer function must be on the upper half complex plane.

Rq: it's needed that $\chi(\omega)$ be analytic for Cauchy theorem to hold.

7. Kramers-Kronig relations

Assume 1) a causal analytic electric susceptibility response $\chi(\omega)$, bounded.
$\Rightarrow$ the poles are in the upper half $\mathbb{C}$ plane.

2) $\lim\limits_{|\omega| \to +\infty} \chi(|\omega|) = 0$



$C_1 : \Omega \qquad -A \leq \Omega \leq \omega - a$

$C_2 : \omega - a e^{j\theta} \qquad 0 \leq \theta \leq \pi$

$C_3 : \Omega \qquad \omega + a \leq \Omega \leq A$

$C_4 : A e^{-j\theta} \qquad 0 \leq \theta \leq \pi$

$$d\Omega = -j\, d\theta\, A e^{-j\theta}$$

Cauchy theorem $\Rightarrow \oint \frac{\chi(\Omega)}{\Omega - \omega}\, d\Omega = 0$

$$0 = \int_{-A}^{\omega - a} \frac{\chi(\Omega)}{\Omega - \omega}\, d\Omega + \int_{\omega + a}^{A} \frac{\chi(\Omega)}{\Omega - \omega}\, d\Omega + \underbrace{\int_0^{\pi} \frac{-jAe^{-j\theta}\, \chi(Ae^{-j\theta})}{Ae^{-j\theta} - \omega}\, d\theta}_{\to 0\ [A \to +\infty]} + \int_0^{\pi} \frac{\cancel{+}j\cancel{a}e^{j\theta}\, \chi(\omega - ae^{j\theta})}{\cancel{\omega}\cancel{+a}e^{j\theta} - \cancel{\omega}}\, d\theta$$

$A \to +\infty$
$a \to 0$
$\Rightarrow$

$$\lim_{a \to 0} \left(\int_{-\infty}^{\omega - a} \frac{\chi(\Omega)}{\Omega - \omega}\, d\Omega + \int_{\omega + a}^{+\infty} \frac{\chi(\Omega)}{\Omega - \omega}\, d\Omega \right) = -j\pi\, \chi(\omega)$$

half a pole contribution

$$\boxed{\chi(\omega) = \frac{j}{\pi}\, P \int_{-\infty}^{+\infty} \frac{\chi(\Omega)}{\Omega - \omega}\, d\Omega}$$

Kramers Kronig relation for causal & bounded response functions

Taking the real and im parts $\chi(\omega) = \chi'(\omega) + j\, \chi''(\omega)$

$$\boxed{\begin{aligned} \chi'(\omega) &= -\frac{1}{\pi}\, P \int_{-\infty}^{+\infty} \frac{\chi''(\Omega)}{\Omega - \omega}\, d\Omega \\ \chi''(\omega) &= \frac{1}{\pi}\, P \int_{-\infty}^{+\infty} \frac{\chi'(\Omega)}{\Omega - \omega}\, d\Omega \end{aligned}}$$

Kramers Kronig relations

8. Consequences of KK relations

1) $\chi(t) \in \mathbb{R} \Rightarrow \chi(-\omega) = \chi^*(\omega)$ crossing condition

$$\Rightarrow \chi'(\omega) = -\frac{1}{\pi} P\int_0^{+\infty} \frac{\chi''(\Omega)}{\Omega-\omega}\, d\Omega + \frac{1}{\pi} P\int_{\infty}^{0} \frac{\chi''(-\Omega)}{-\Omega-\omega}\, d\Omega$$

$$= -\frac{1}{\pi} P\int_0^{+\infty}\left[\frac{\chi''(\Omega)}{\Omega-\omega} + \frac{\chi''(\Omega)}{\Omega+\omega}\right] d\Omega \Rightarrow \boxed{\chi'(\omega) = -\frac{2}{\pi} P\int_0^{+\infty} \frac{\Omega\chi''(\Omega)}{\Omega^2-\omega^2}\, d\Omega}$$

$$\Rightarrow \left.\frac{\partial \chi'}{\partial \omega}\right|_{\omega_1} = -\frac{2}{\pi}\frac{\partial}{\partial \omega}\int_0^{+\infty} \frac{\Omega\chi''(\Omega)}{\Omega^2-\omega^2}\, d\Omega$$

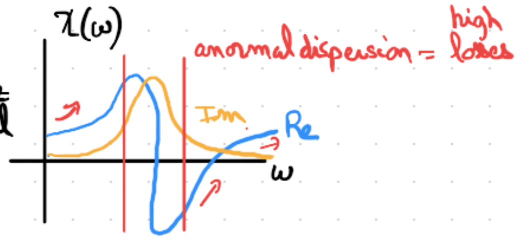
Assuming P goes away because lossless $(\chi''(\omega)=0)$ around ω_1

$$= -\frac{2}{\pi}\int_0^{+\infty} \frac{2\omega\Omega\overbrace{\chi''(\Omega)}^{<0 \text{ if passive}}}{(\Omega^2-\omega^2)^2}\, d\Omega$$

$$\Rightarrow \left.\frac{\partial \chi'}{\partial \omega}\right|_{\omega_1} > 0 \Rightarrow \boxed{\frac{\partial \operatorname{Re}[\varepsilon]}{\partial \omega} > 0}$$

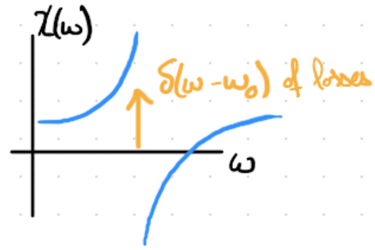
normal dispersion
PASSIVE
CAUSAL
LOSSLESS region of spectrum

Lorentz model



$\gamma \to 0$

$\Rightarrow$



$$\varepsilon'' = -\frac{\pi}{2}\frac{\omega_p}{\omega_0}\delta(\omega-\omega_0) \overset{KK}{\Rightarrow} \varepsilon' = \varepsilon_\infty + \frac{\omega_p^2}{\omega_0^2-\omega}$$

2. $\chi'(\omega)$ is difficult to measure in optics (phase) but absorption is easy

$\Rightarrow$ measure absorption $\Rightarrow \chi''(\omega) \Rightarrow \chi'(\omega)$

Same with the reflection phase $r = \rho e^{j\theta} \Rightarrow \ln r = \ln\rho + j\theta$ is analytic

for $z>0$, $\pi \leq \theta < 2\pi$, causal but $\lim_{|\omega|\to+\infty} \ln r(|\omega|)$? $\Rightarrow$ apply KK to $\frac{\ln r}{\omega}$

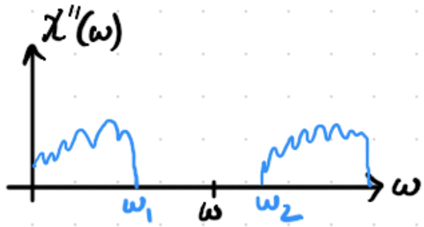
after a few calculations

$$\Rightarrow \boxed{\underset{\text{deduce}}{\theta(\omega)} = \frac{-2\omega}{\pi} P\int_0^{+\infty} \frac{\overset{\text{measure}}{\ln\rho(\Omega)}}{\Omega^2-\omega^2}\, d\Omega} \Rightarrow \text{deduce complex } n$$

3- Vacuum is the only possible completely lossless medium:

$$\chi''(\Omega) = 0 \ \forall\Omega \Rightarrow \chi'(\omega) = 0 \ \forall\omega \Rightarrow \varepsilon(\omega) = \varepsilon_0 \ \forall\omega$$

4. Transparency window



$$\mathrm{Re}\left[\frac{\varepsilon(\omega)}{\varepsilon_0}\right] = 1 - \frac{2}{\pi}\int_0^{\omega_1} \frac{\Omega\,\mathrm{Im}[\varepsilon(\Omega)/\varepsilon_0]}{\cancel{\Omega^2} - \omega^2}\,d\Omega - \frac{2}{\pi}\int_{\omega_2}^{+\infty} \frac{\Omega\,\mathrm{Im}[\varepsilon(\Omega)]/\varepsilon_0}{\Omega^2 - \cancel{\omega^2}}\,d\Omega$$

when $\omega_1 \ll \omega \ll \omega_2$

$$\Rightarrow \mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \underbrace{\frac{2}{\pi}\int_{\omega_2}^{+\infty} \frac{\overbrace{\mathrm{Im}\left[\frac{\varepsilon(\Omega)}{\varepsilon_0}\right]}^{<0}}{\Omega}}_{\varepsilon_\infty \,=\, \text{const}} + \underbrace{\frac{2}{\pi\omega^2}\int_0^{\omega_1} \Omega\,\mathrm{Im}\left[\frac{\varepsilon(\Omega)}{\varepsilon_0}\right] d\Omega}_{-\frac{\omega_p^2}{\omega^2}} \Rightarrow \text{Drude region!}$$

concentrated losses $\Rightarrow$ Lorentzian resonance

transparency window $\Rightarrow$ Drude region

$\varepsilon_\infty > 1$ means losses are present at higher frequency

(5 for Ag at optical frequencies)

ω_p depends on losses at low frequencies

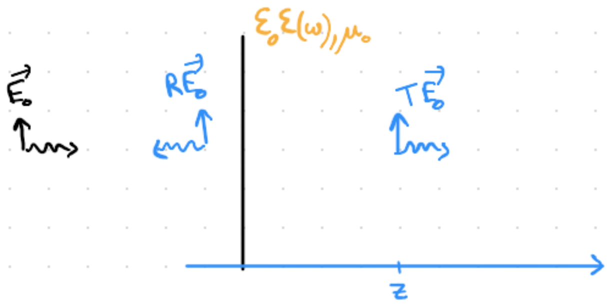
5. Sum rule and static limit

$$\mathrm{Re}\left[\varepsilon(\omega=0)\right] = \varepsilon_s = \varepsilon_0 - \frac{2}{\pi}\int_0^{+\infty} \frac{\mathrm{Im}(\varepsilon)}{\Omega}\,d\Omega > \varepsilon_0$$

if no losses at $\omega = 0$ (dielectric)

$\boxed{\varepsilon_s > \varepsilon_0}$ large $\varepsilon_s \Rightarrow$ losses at high f

9. Time of arrival



time to arrive to z ?

<u>Normal incidence</u> : $R = \frac{\eta - \eta_0}{\eta + \eta_0} = \frac{\sqrt{\frac{1}{\varepsilon}} - 1}{\sqrt{\frac{1}{\varepsilon}} + 1} = \frac{1-n}{1+n}$ $T = 1 + R = \frac{2}{1+n(\omega)}$ $n(\omega) = \sqrt{\varepsilon}$

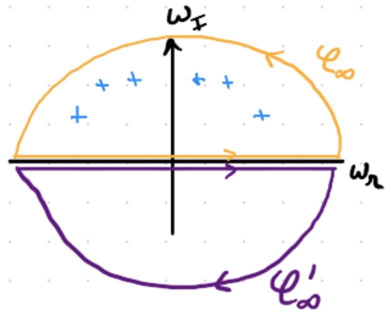
$$\vec{E}_0(z=0,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_0(\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow \vec{E}_0(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_0(\omega) \cdot \underbrace{\frac{2}{1+n(\omega)}}_{\text{causal, bounded}}\, \overbrace{e^{-j \underbrace{k(\omega)}_{\frac{\omega n(\omega)}{c}} z}\, e^{j\omega t}}^{\exp\left[-j \frac{\omega}{c}(n(\omega) z - ct)\right]}$$

Assuming a transparency window with $\omega_2 \to +\infty$: $\mathrm{Im}\left(\frac{\varepsilon}{\varepsilon_0}\right) \to 0$ $[|\omega| \to +\infty]$

$$\mathrm{Re}\left(\frac{\varepsilon}{\varepsilon_0}\right) = 1 + \frac{2}{\pi\omega^2} \int_0^{\omega_1} \Omega\, \mathrm{Im}\left(\frac{\varepsilon}{\varepsilon_0}\right) d\Omega = 1 - \frac{\omega_p^2}{\omega^2} \Rightarrow$$

$n(|\omega|) \to 1$ $[|\omega| \to +\infty]$ $\Rightarrow$ forget $n(\omega)$ on $\mathcal{C}_\infty$ or $\mathcal{C}'_\infty$



$\forall (z - ct) > 0$, $e^{-j \frac{\omega_R}{c}(z-ct)}\, e^{\frac{\omega_I}{c}(z-ct)}$

makes the integral vanish on $\mathcal{C}'_\infty$

$\Rightarrow \vec{E}_0(z,t) = 0 \Rightarrow$ <u>nothing can arrive faster than c</u> if KK relations hold *

$z - ct < 0 \Rightarrow$ integral vanishes on $\mathcal{C}_\infty$ but poles $\Rightarrow$ receives signal.

* We also assumed a transparency window at ∞ frequency and $\mu = \mu_0$.
See more about this in Jackson's book, ch 7 and references therein.

* <u>phase velocity</u> = speed of the phase of a plane wave = no causality involved

$$v_{ph} = \frac{\omega}{k} \qquad 0 \leq v_p \leq +\infty \quad \text{can be superluminal}$$

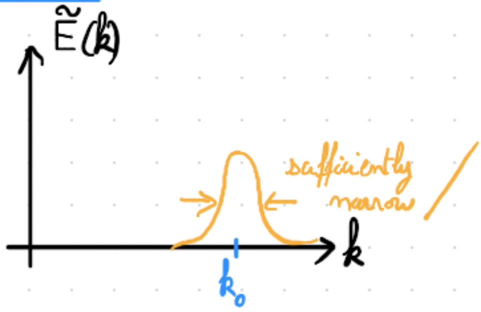
* <u>group velocity</u> = speed of a wavepacket

Let's assume the following pulse at $t=0$: $E_y(z,0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{E}(k)\, e^{-jkz}\, dk$ arbitrary

$\tilde{E}(k) = \int_{-\infty}^{+\infty} E_y(z,0)\, e^{+jkz}\, dz$ is the spatial Fourier transform at k

Later: $E_y(z,t) = \frac{1}{2\pi} \mathrm{Re} \int_{-\infty}^{+\infty} \tilde{E}(k)\, e^{-jkz}\, e^{j\omega t}\, dk$ with $\omega = \omega(k)$ → hard integral in general

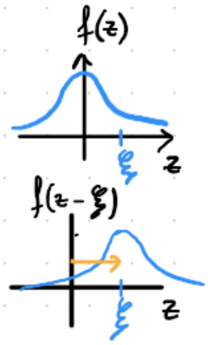
<u>Assume</u>:



$$\Rightarrow \omega(k) = \overbrace{\omega(k_0)}^{\omega_0} + \left.\frac{\partial \omega}{\partial k}\right|_{k_0} (k - k_0)$$

$$\Rightarrow E_y(z,t) = \frac{1}{2\pi} \mathrm{Re} \int_{-\infty}^{+\infty} \tilde{E}(k)\, e^{-jkz}\, e^{j\omega_0 t}\, e^{j \left.\frac{\partial\omega}{\partial k}\right|_{k_0} (k-k_0) t}\, dk$$

$$= \frac{1}{2\pi} \mathrm{Re} \left\{ e^{j\left(\omega_0 - \left.\frac{\partial\omega}{\partial k}\right|_{k_0} k_0\right) t} \int_{-\infty}^{+\infty} \tilde{E}(k)\, e^{-jk\left(z - \left.\frac{\partial\omega}{\partial k}\right|_{k_0} t\right)} \right\}$$



$$\Rightarrow E_y(z,t) = \mathrm{Re} \left\{ e^{j\left(\omega_0 - \left.\frac{\partial\omega}{\partial k}\right|_{k_0} k_0\right) t}\, E_y\Big(z - \underbrace{\left.\frac{\partial\omega}{\partial k}\right|_{k_0} t}_{\xi \text{ is linear with } t}, 0\Big) \right\}$$

The pulse propagates at $v_{gr} = \frac{\partial \omega}{\partial k}$ $\qquad \boxed{0 \leq v_{gr} \leq c}$

Generalization to 3D $\boxed{\vec{v}_{gr} = \vec{\nabla}_k\, \omega}$

* energy velocity : $\boxed{v_{en} = \dfrac{\vec{S}\cdot\vec{z}}{W_e + W_m}}$ $\left[\dfrac{W/m^2}{J/m^3} \equiv \dfrac{m}{s} \right]$

$\vec{S}$ Poynting vector at ω_0

$\vec{z}$ direction of propagation (plane wave)

W_e, W_m : electric and magnetic stored energy densities at ω_0

$$\boxed{W_e = \frac{1}{4} \underbrace{\frac{\partial \omega \varepsilon}{\partial \omega}}_{>0} |\vec{E}|^2 \qquad W_m = \frac{1}{4} \underbrace{\frac{\partial \omega \mu}{\partial \omega}}_{>0} |\vec{H}|^2}$$

only meaningfull in low loss spectral regions

* examples for a Drude material $k = \omega\sqrt{\mu_0 \varepsilon(\omega)} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$

* $v_{ph} = \dfrac{\omega}{k} = \dfrac{\omega}{\frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}}} = \dfrac{c}{\sqrt{1-\frac{\omega^2}{\omega_p^2}}}$ $\omega > \omega_p \Rightarrow v_{ph} > c$

* $v_{gr} = \dfrac{\partial \omega}{\partial k} = ?$ $c^2 k^2 = \omega^2\left(1 - \frac{\omega_p^2}{\omega^2}\right) = \omega^2 - \omega_p^2 \Rightarrow \cancel{2} c^2 k\,dk = \cancel{2}\omega\,d\omega$

$\Rightarrow v_{gr} = c^2 \dfrac{k}{\omega} = \dfrac{c^2}{v_{ph}} = c\sqrt{1 - \dfrac{\omega_p^2}{\omega^2}} = v$ $v_{gr} < c$ always

* $v_{en} = \dfrac{\vec{S}\cdot\vec{z}}{\omega_e + \omega_H} = \dfrac{\frac{1}{\cancel{2}}\,|\vec{E}|^{\cancel{2}}/\eta_0 \cdot \sqrt{1 - \frac{\omega_p^2}{\omega^2}}}{\frac{1}{\cancel{4}_{2}}\underbrace{\mu_0 \frac{|\vec{E}|^{\cancel{2}}}{\eta_0^2}}_{\varepsilon_0}\left(1 - \cancel{\frac{\omega_p^2}{\omega^2}}\right) + \frac{1}{\cancel{4}_{2}}\underbrace{\frac{\partial}{\partial \omega}\left(\omega - \frac{\omega_p^2}{\omega}\right)}_{1 + \cancel{\omega_p^2/\omega^2}}\varepsilon_0 |\vec{E}|^{\cancel{2}}}$

$\Rightarrow v_{en} = \dfrac{1}{\eta_0}\sqrt{1 - \dfrac{\omega_p^2}{\omega^2}}\Big/\varepsilon_0 = \boxed{c\sqrt{1 - \dfrac{\omega_p^2}{\omega^2}} = v_{en}}$

Rk 1 : $\boxed{\dfrac{c^2}{v_{ph}\, v_{gr}} = 1}$ so beautiful

it speaks for itself

Rk 2 : $\boxed{v_{gr} = v_{en}}$ for all Drude materials (v_{en} is undefined anyways when losses can't be neglected $\Rightarrow$ very general results)